Mathematical Caring Relations in Action

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In a small-scale, 8-month teaching experiment, the author aimed to establish and maintain mathematical caring relations (MCRs) (Hackenberg, 2005c) with 4 6th-grade students. From a teacher’s perspective, establishing MCRs involves holding the work of orchestrating mathematical learning for students together with an orientation to monitor and respond to energetic fluctuations that may accompany student–teacher interactions. From a student’s perspective, participating in an MCR involves some openness to the teacher’s interventions in the student’s mathematical activity and some willingness to pursue questions of interest. In this article, the author elucidates the nature of establishing MCRs with 2 of the 4 students in the study and examines what is mathematical about these caring relations. Analysis revealed that student–teacher interaction can be viewed as a linked chain of perturbations; in student–teacher interaction aimed toward the establishment of MCRs, the linked chain tends toward perturbations that are bearable (Tzur, 1995) for both students and teachers.

Key words: Affect; Constructivism; Fractions; Learning; Middle grades, 5–8; Reasoning; Teaching (role, style, methods)

In Noddings’s (2002) care theory, caring is conceived of not primarily as a characteristic of a person such as a teacher, but as an interaction—an evolving relationship—in which both teacher and student participate. The foundation of caring relations is the teacher’s reception of students, in which receiving is not motivated by reason but by a desire to maintain relatedness. However, caring relations are not established unless there is some evidence that the student receives the teacher’s care. Such evidence includes a renewed interest in an activity, an increase in energy level, or even a “glow of well-being” (p. 28). In turn, this reception of care is what a teacher needs to feel cared for—to also experience renewed interest or energy. Thus, establishing caring relations involves fostering experiences of relatedness for both teachers and students.

Indeed, psychologists posit that relatedness is a fundamental human need (Baumeister & Leary, 1995; Ryan & Deci, 2000), the satisfaction of which influences the level of positive energy that a person experiences as available to the self, or a person’s subjective vitality (Ryan & Frederick, 1997). The enhancement of
subjective vitality has been associated with both physical and psychological well-being (Kasser & Ryan, 1999; Muraven, Gagné, & Rosman, 2008; Ryan & Deci, 2008; Ryan & Frederick, 1997), whereas the diminishment of it is detrimental to health and cognitive performance (Baumeister, Bratslavsky, Muraven, & Tice, 1998; Baumeister & Vohs, 2007; Thayer, 2001). Although Noddings does not refer to evidence of the reception of care in terms of subjective vitality, this concept of energy available to the self seems compatible with her formulation of renewed energy as a central indicator of receiving care.

In this article, I build on Noddings’s caring relations and the construct of subjective vitality to define and study what I call mathematical caring relations (MCRs). I define a mathematical caring relation (MCR) as a quality of interaction between a student and a teacher that conjoins affective and cognitive realms in the process of aiming for mathematical learning (Hackenberg, 2005c). Mathematics teachers act as carers in general, but they start to act as mathematical carers when they hold their work of orchestrating mathematical learning for their students together with an orientation to monitor and respond to fluctuations in positive energy available to the self—subjective vitality—that may accompany student–teacher interactions. From a student’s perspective, participating in an MCR with a teacher involves some openness to the teacher’s interventions in the student’s mathematical activity and some willingness to pursue questions and ideas of interest, even if these questions and interests are not initially valued or anticipated by the student.

This characterization of MCRs was the basis for establishing these relations with four sixth-grade students in an 8-month constructivist teaching experiment. The purpose of the experiment was to study algebraic aspects of students’ quantitative reasoning in the context of a teacher who endeavored to establish MCRs with them. In this article I focus on two of the four students and address two questions: What is the nature of establishing MCRs in extensive, small-scale interaction between students and their teacher (the researcher and author)? What is mathematical about the caring relations?

STUDYING STUDENT–TEACHER RELATIONSHIPS

Prior research on relationships between students and teachers has used statistical methods to uncover positive correlations between assessments of these relationships and other constructs such as student achievement and engagement (Cornelius-White, 2007; O’Connor & McCartney, 2007; Pianta, 1999), self-esteem (Ryan, Stiller & Lynch, 1994), autonomy (Ryan & Grolnick, 1986), and motivation (Midgley, Feldlaufer, & Eccles, 1989; Murdock & Miller, 2003; Patrick, Ryan, & Kaplan, 2007; Wentzel, 1997). Although some of this research has been longitudinal (e.g., O’Connor & McCartney), most of it has not focused on how student–teacher relationships form, but instead on the possible influences these relationships may have on students, usually at the general level of schooling rather than in a particular subject domain. For example, the findings from O’Connor and McCartney’s study of the changing quality of student–teacher relationships from
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preschool through third grade upheld the hypothesis that strong student–teacher relationships positively influence student achievement by enhancing student engagement in classrooms. In addition, in Wentzel’s study, middle school students’ perceptions of their teachers as caring were positively related to academic effort and the pursuit of prosocial goals.

An exception to the focus on schooling in general is the research of Midgley and colleagues. These researchers found that when elementary school students with teachers who they viewed as low in support moved to a middle school mathematics teacher they perceived to be high in support, the students experienced an enhanced appreciation of the intrinsic value of mathematics. However, little is known about how the nurturance of student–teacher relationships may be involved in the process of mathematical learning. Researchers studying the processes of mathematical learning have debated the role of human (social) interaction (e.g., Cobb, Boufi, McLain, & Whitenack, 1997; Cobb, Jaworski, & Presmeg, 1996; Lerman, 1996, 2000; Steffe, 1996, 2003; Steffe & Thompson, 2000a; van Oers, 1996; Wood, Williams, & McNeal, 2006). But these debates have not focused on how student–teacher relationships form, or how they contribute to mathematical learning. In fact, the development and influence of student–teacher relationships—and, in particular, caring relations between students and teachers—has been understudied in mathematics education (Vithal, 2003).

Care theory in general, and Noddings’s (1984, 2002) caring relations in particular, has been used to study teacher caring, especially with regard to teacher preparation (e.g., Goldstein & Lake, 2000; Philipp et al., 2007; Philipp, Thanheiser, & Clement, 2002; Rogers & Webb, 1991). For example, Philipp and colleagues have developed the notion of “circles of caring” (2002, p. 198) to describe how they capitalize on the care that prospective elementary teachers feel for children, using it as a critical entry point for developing prospective teachers’ care for children’s mathematical thinking and for mathematics. Other studies of care between teachers and students have focused on qualities of teachers that students perceive as caring (e.g., Alder, 2002; Hayes, Ryan, & Zseller, 1994; Wentzel, 1997), on whether teacher caring influences student motivation and perceived achievement (e.g., Ding & Hall, 2007; Murdock & Miller, 2003), or on the work involved in teacher caring (e.g., Acker, 1995; Teven, 2007). Consistent with the research on student–teacher relationships, few studies on teacher caring examine how caring relationships between students and teachers form.

This study contrasts with prior research on student–teacher relationships and teacher caring in three ways. First, in this study I used qualitative methods to examine the development of student–teacher relationships over an extended (8-month) period of time, trying to understand how such relationships form. So, in contrast with prior research, I sought to study the phenomenological act of building a caring relationship (Van Manen, 2000). Second, I aimed to understand student–teacher relationships in a particular subject domain and how these relationships may be involved in the process of learning in that domain. My aim was not strict “isolation” of mathematical caring from general caring, but rather exploration and
refinement of what it means to establish a caring relation between a teacher and student engaged in long-term interaction aimed toward mathematical learning. Third, I included a focus on affective “outcomes” for the teacher, not just on cognitive and affective outcomes for students. Thus, the study required considering how caring relations between students and teachers are initiated and maintained, how mathematical learning happens, and how energy levels fluctuate in building relationships. Toward this end, I frame the article using three bodies of literature: care theory, scheme theory, and research on subjective vitality.

CARE THEORY, MATHEMATICAL LEARNING, AND SUBJECTIVE VITALITY

Caring Relations

In Noddings’s (2002, 2003, 2005) notion of caring relations, a teacher establishes care for students by listening intently to students’ ideas and receiving their experiences, desires, thoughts, and worlds with “nonselective attention” (2005, p. 91). Doing so does not mean just projecting the teacher’s own reality onto the student, but instead requires the teacher to consider what it would be like to act or think in the way the student does (2003). In other words, a teacher-as-carer decenters from his or her own perspectives and marshals energy to help students realize and expand their ideas and worlds, just as the teacher might work to realize and expand his or her own ideas and world. As I have indicated, Noddings characterizes the experience of receiving care in terms of increased energy. Evidence of the reception of the teacher’s care is what the teacher needs most to feel “cared for” by the student and to continue to care—and in turn to enhance the teacher’s levels of energy. Thus, for Noddings, caring is not just a feeling; caring signals an effort to cocreate and participate in social interactions that are responsive to the needs of both teachers and students.

In fact, Noddings (2005) proposes that the first purpose of schooling should be to examine and establish care for self, other people, animals, objects, instruments, and ideas. She exhorts adults to be educators first and teachers of particular subjects, such as mathematics, second (see also Noddings, 1986). Although I do not disagree with Noddings’s general arguments or sentiments, in this article I show how establishing and maintaining a caring relation aimed toward mathematical learning with a student—an MCR—requires considerable attention to cognition. How a teacher interprets the mathematical thinking of her students has a significant influence on the degree to which the teacher can care mathematically for the students in ongoing interaction with them. Interpreting the mathematical thinking of students in such a way that it can form a basis for their learning cannot be accomplished solely by nonselective attention to them, or by marshaling energy to help them realize and expand their experiences. My study indicates that cognitive decentering is a crucial part of establishing a caring relation that is mathematical.

I view cognitive decentering as the practice of setting one’s own concepts to the side in order to formulate a coherent description of the mental activity of a student
that can be the basis for interaction with that student (Confrey, 1998; Steffe & Thompson, 2000b). Cognitive decentering goes beyond just knowing that a student thinks differently to attempting to think like the student thinks, and acting upon that attempt to open possibilities for the student to make progress in some way (Thompson, 2000). Efforts to decenter cognitively will vary in nature from intuitive responsiveness to a student’s thinking to a more deliberate, analytical formulation of how to interact, and often a combination of intuitive and analytical activity is required.

Mathematical Learning

To practice cognitive decentering in mathematical interaction requires a theory of mathematical learning. Following Piaget (1970, 1971) and von Glasersfeld (1995), I view mathematical learning in the context of making accommodations in schemes and operations. Operations are mental actions such as dividing a unit (whole) into parts (partitioning), or repeatedly instantiating a unit to create a plurality of units (iterating). Operations are the components of schemes: goal-directed ways of operating that consist of a perceived situation, an activity, and a result (Piaget, 1971; von Glasersfeld, 1995).1 An accommodation may occur when a person’s current schemes and operations produce an unexpected result: The person does not achieve her intended goal. This “disturbed” state of affairs is one example of a perturbation and is often accompanied by a sense of disappointment or surprise. That is, a perturbation often yields an emotion.

This observation is consistent with the views of researchers in the area of emotions and education, such as Schutz, Pekrun, and colleagues (Pekrun, 2006; Pekrun, Goetz, Titz, & Perry, 2002; Schutz & DeCuir, 2002; Schutz & Pekrun, 2007), who take an emotion to be one outcome of a person’s appraisal of the extent to which a person’s goal is being met (Lazarus, 1991).2 Since a perturbation necessarily involves an appraisal (not necessarily at a conscious level) that a goal is not being met; it is likely to trigger an emotion. Both cognitive appraisals of meeting goals and the experience of emotions are accompanied by variations in the positive energy one experiences as available to the self, that is, in one’s subjective vitality (Ryan & Deci, 2008; Ryan & Frederick, 1997; cf. Thayer, 2001).3 So, another outcome of a perturbation is a person’s level of subjective vitality. I will refer to these “outcomes” as emotional and energetic responses to perturbations, even though they may be experienced as intertwined with perturbations, not necessarily as following them.

1I use the phrase ways of operating to refer to a range of repeatable activity in which a person engages, such as a student regularly telling the teacher that his or her head hurts when a problem seems difficult, or a student consistently stating reciprocal multiplicative relationships between two quantities. I use schemes and operations in the more specific sense defined here.

2Note that genetic dispositions and physiological processes are also major sources of emotions (Pekrun et al., 2002).

3For Thayer (2001), energy levels and patterns of tension underlie emotions and moods.
The emotional and energetic response to a perturbation is a major point of connection between learning and caring, and one that has received little attention in mathematics education research. Depletion of subjective vitality may occur if a person senses that she or he does not know what to do to eliminate the perturbation, or that such activity will be particularly onerous. If depletion is too great or extended for too long, a student may feel overwhelmed, which may impede engagement in mathematical activity immediately or in the future (Tzur, 1995). Perturbations can also provoke an enhancement of subjective vitality if they are experienced as a challenge, particularly if a person senses that she or he can meet that challenge, or that such activity itself will be enjoyable. If the level of subjective vitality is sufficient, the student’s interest in or curiosity about a situation may prolong mathematical activity and open new opportunities for learning. If, over time, subjective vitality is enhanced, even though there may be periods of depletion, the student may feel mathematically cared for.

Subjective Vitality as a Marker of Receiving Mathematical Care

Subjective vitality lies in the affective realm since it is a psychological experience of energy and aliveness, characterized by feelings of enthusiasm and vigor (Nix, Ryan, Manly, & Deci, 1999; Ryan & Deci, 2008; Ryan & Frederick, 1997) or calm alertness (Thayer, 2001). Ryan and Deci (2008) have posited that subjective vitality will be maintained or enhanced by satisfying basic human needs for autonomy, competence, and relatedness. Much of the research on subjective vitality thus far has focused on how fostering or thwarting autonomy may affect subjective vitality (e.g., Baumeister, et al., 1998; Baumeister & Vohs, 2007; Moller, Deci, & Ryan, 2006; Muraven, Gagné, & Rosman, 2008; Nix et al., 1999). Less explicit attention has been paid to the relationship between relatedness and subjective vitality (Gagné, Ryan, & Bargmann, 2003; Kasser & Ryan, 1999), in which relatedness is considered to be “feeling significant and connected” (Ryan & Deci, 2008, p. 711). To my knowledge, no research in mathematics education has made use of subjective vitality to examine relatedness in interactions between students and mathematics teachers.

The construct of subjective vitality can be used as a marker or indicator of the reception of mathematical care, since it helps elaborate Noddings’s (2002) notion of energy or “glow of well-being” (p. 28) as what a student who receives mathematical care experiences. In particular, the enhancement of subjective vitality can manifest as either a positive aroused state such as excitement or enjoyment (Ryan & Deci, 2008; Ryan & Frederick, 1997), or as calm energy as conceived of by Thayer (2001). Calm energy is a state of high energy combined with feelings of relaxation or low tension, as opposed to other states of high energy and high tension (tense energy), low energy and low tension (calm tiredness), or low energy and high tension (tense tiredness). Thayer’s construct of calm energy indicates that overt arousal is not the only way to exhibit an increase in vitality, and that tension is a factor that contributes to depletion of energy.
Prior to the study, I used Noddings’s (2002) care theory and my theory of learning based on von Glasersfeld’s (1995) constructivism to postulate what teachers need to do to establish MCRs with students. This theoretical formulation was a starting place to guide my actions during the experiment. To establish MCRs, a teacher attempts to pose problems in which students may construct more powerful schemes while experiencing a balance between enhancement and depletion of subjective vitality. To accomplish this goal, teachers pose problems that harmonize with students’ current schemes and energetic responses to mathematical activity, as well as problems that challenge students—that open opportunities for them to make accommodations and thereby develop their mathematical ways of operating. Throughout this process, teachers track students’ energetic and cognitive responses to mathematical activity, reinitiate harmonizing and challenging, and monitor their own levels of subjective vitality.

Harmonizing with students’ schemes and energetic responses to mathematical activity requires modeling students’ mathematical thinking and energetic responses well, and using those models as a basis for interaction with students. Following Steffe and colleagues (Steffe & Thompson, 2000a; Steffe, von Glasersfeld, Richards, & Cobb, 1983; Thompson, 2000), I take first-order models to consist of a constellation of constructs a person builds up to “order, comprehend, and control his or her experience” (Steffe et al., 1983, p. xvi). A second-order model is a constellation of constructs a teacher–researcher formulates of a student’s thinking to account for the teacher–researcher’s experience of the student’s states and activities (Steffe et al., 1983, p. xvi). Making good second-order models of students’ thinking and energetic responses to mathematical activity is critical for harmonizing with students’ schemes and energetic responses, because the models are a central tool with which to interact with students—the models orient a teacher–researcher to what will be sensible and amenable to the students.

However, the description of harmonizing as “simply” good second-order model-building elides the dynamism involved. Harmonizing indicates that the teacher–researcher is monitoring interaction with students and making adjustments in response to the teacher–researcher’s continually changing understanding about the students. Noddings (2001) says that “. . . every claim to care must eventually be grounded in the response of the cared-for . . .” (p. 102). I interpret this statement to mean that the carer evaluates his or her efforts to care in relation to evolving interpretations of the cared-for’s responses to those efforts. In characterizing harmonizing, I borrow this idea to mean that a teacher who harmonizes with students’ schemes and energetic responses to mathematical activity is focused on making in-the-moment interpretations of both parties’ contributions to the interaction as it proceeds, grounding any claim of harmonizing in the students’ cognitive and energetic responses to those efforts.

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4If a new scheme allows a learner to solve problems not previously solved and can serve in further learning, then it can be considered more powerful than a previous scheme.
METHOD

Teaching Experiment Methodology

Constructivist teaching experiment methodology (Confrey & Lachance, 2000; Steffe & Thompson, 2000b) is well suited for studying the establishment of MCRs with students because researchers aim to think like students think and to open opportunities for students to construct new schemes. A central goal of this kind of teaching experiment is to understand and explain how students operate mathematically and how their ways of operating change in the context of teaching. Because researchers’ mathematical ways of operating may be insufficient to understand students’ ways of operating, researchers aim to learn mathematical ways of operating from students. Researchers also engage in ongoing conceptual analysis of how students might operate in mathematical situations. Based on learning from students and conceptual analysis, researchers make conjectures and test them through posing tasks in teaching episodes. Teaching practices include presenting students with problems, analyzing students’ responses, and determining new problems that might allow students to make accommodations.

Witness–researchers play a critical role in teaching experiments by providing other perspectives during all three phases of the experiment: the actual teaching episodes, ongoing analysis that occurs between teaching episodes, and retrospective analysis (Steffe & Thompson, 2000b). While observing the teaching episodes, witness–researchers may formulate questions to test a particular conjecture that the teacher–researcher, enmeshed in the interaction, cannot immediately see. Witness–researchers usually pass their questions on to the teacher–researcher; if they want to ask a question directly, they ask permission of the teacher–researcher. So, the teacher–researcher can always assess whether or not the question is appropriate in the moment, given the teacher–researcher’s goals at that moment. Witness–researchers also provide valuable triangulation of interpretations in both ongoing and retrospective analysis, which I discuss later in this section.

To study MCRs, I adapted this methodology so that attention to both cognitive and energetic responses influenced how I intervened in interactions with the participating students. In the moment of interaction in this kind of teaching experiment, a researcher aims to pose tasks (often spontaneously) that will allow him or her to construct a working second-order model of students’ zones of potential construction (ZPCs) (Norton & D’Ambrosio, 2008; Steffe & D’Ambrosio, 1995; Steffe & Thompson, 2000b). In my adaptation of this methodology, I posed tasks aimed not only at constructing a working model of the students’ mental activity but also at generating a model of their energetic responses to our mathematical interaction. In other words, I was invested not exclusively in constructing the

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5The zone of potential construction of a student is a time-sensitive concept describing ways of operating that a student might construct based on the student’s current schemes and operations (Steffe & D’Ambrosio, 1995).

6From this point onward, I use “model” to refer to second-order models.
students as thinkers but also in constructing them as affective beings. This aim meant that I was sensitive to their energy levels and might curtail our activity if depletion seemed too great, or prolong our activity in some way (often by extending problems or posing what I thought were greater challenges) if the students appeared to be experiencing a balance between depletion and enhancement. In short, I used subjective vitality as a marker or indicator of the reception of my mathematical care, or the experience of a lack of mathematical care.

In addition, I decided against asking students explicitly about their energetic responses to our interactions for two reasons. First, I was interested in studying energetic responses, and the management of those responses, in the moments of interaction. Energetic responses to interactions are not always within conscious awareness, so it may not be possible to make an assessment of students’ energetic states by asking them to articulate their responses. Second, I was not interested in studying how students hold out their energetic responses for inspection, but in the energetic flow as the student–teacher interaction proceeded. I concluded that explicit questioning could interfere with this flow. In this way, my research differs from research focused explicitly on beliefs and attitudes, which are conceived of as relatively stable and are often assessed through more overt questioning (Lazarus, 1991; McLeod, 1992).

Finally, I want to emphasize that, consistent with this kind of constructivist teaching experiment methodology, I was trying to create a situation that I could then analyze. So I aimed to influence students’ mathematical learning and energetic responses in “positive” ways, that is, to engender the most possible learning with the students and to facilitate the students’ experiences of vital energy. My knowledge of this goal necessarily influenced what I observed—I was a part of the intervention that I then analyzed. In addition, even though I conducted the study focused on the vital energy and energetic responses of the students and myself, I did not use the term subjective vitality until after the conclusion of the experiment, because I became aware of Ryan and Frederick’s (1997) work in this area after data collection was completed.

**My Teaching Experiment**

Four sixth-grade students from a rural middle school in Georgia participated in my teaching experiment from October to May. They were invited to participate after demonstrating during unrecorded selection interviews that they were reasoning multiplicatively with whole numbers (Hackenberg, 2005a). I taught them twice per week in pairs for 2 to 3 weeks, followed by 1 week off.7 So over the 8 months, I met with each pair approximately 35 times. Each teaching episode occurred at the school during school hours, lasted approximately 30 minutes, and was videorecorded with two cameras. One camera captured the activity of the students on paper  

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7I chose to work with the students in pairs for two reasons: (a) Students might feel more comfortable talking with their partner than with me, especially initially, and (b) I believed I could learn more about the establishment of MCRs from potential contrasts between partners.
or computer, and the other focused on the interaction between the students and the teacher. Two witness–researchers assisted in videotaping, offered feedback on the teaching activities, and provided triangulation of perspectives in ongoing and retrospective data analysis. In most teaching episodes, the students worked with computer software called JavaBars (Biddlecomb & Olive, 2000), which is a micro-world designed to foster students’ construction of key operations in building fraction schemes.

A major focus of the teaching experiment was to understand how the students solved problems that involved multiplicative relationships between two quantities. For example, consider this problem:

The CD Problem. My stack of CDs is 65 cm tall, and my stack is five times taller than yours. Can you draw a picture of this situation? How tall is your stack?

To solve the CD Problem students need to posit the height of a stack that can be repeated five times to make the known quantity, rather than repeat the 65-cm stack height five times. Doing so requires conceptual insight that is not required by a problem in which students are to repeat a known quantity some number of times (Hackenberg, in press; Norton, 2008; Steffe, 2002). From my perspective, solving the CD Problem requires reasoning with the reverse of the stated multiplicative relationship, so I call the CD Problem and others like it reversible multiplicative relationship (RMR) problems. More complex RMR problems involve quantities and relationships that are fractions:

The Box Problem. Two groups of students, the Cobras and Lizards, have a box-stacking contest. The Cobras’ tower is 3/4 of a decameter tall, and that’s 2/3 of the height of the Lizards’ tower. Can you draw a picture of this situation? How tall is the Lizards’ tower?

To solve the Box Problem without using an algebraic equation or recourse to a standard computational algorithm requires determining how to make a “whole” given a fractional part of it. Doing so relies on reversing one’s reasoning with fractions (Hackenberg, in press; Steffe & Olive, in press; Tzur, 2004).

By November, all four students solved problems like the CD Problem and slightly more complex RMR problems such as the following:

The Money Problem. Tanya has $16, which is 4/5 of what David has. Can you draw a picture of this situation? How much money does David have?

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8 One witness–researcher was another graduate student in mathematics education; the other witness–researcher was a mathematics education professor with significant experience in teaching experiment methodology and research on students’ mathematical learning.

9 JavaBars is a Java version of TIMA: Bars, one of the three computer microworlds developed as part of the teaching experiment Children’s Construction of the Rational Numbers of Arithmetic, which took place from 1990–1995 at University of Georgia under the direction of Leslie P. Steffe and John Olive.
The Money Problem is more complex than the CD Problem because it involves a relationship that is a fraction. However, the numerator of this relationship divides the whole number quantity, which makes it an easier problem than if Tanya’s $16 were, say, 3/5 of David’s money. Only two of the four students constructed schemes for solving RMR problems like this more complex variation of the Money Problem, as well as problems like the Box Problem (Hackenberg, in press). In this article I focus on one of these students and one student who did not solve the more complex RMR problems.

Data Analysis

Ongoing

In constructivist teaching experiments, ongoing analysis occurs in between teaching episodes. The main goal of ongoing analysis is to formulate working models of the mathematical thinking of the students, and in my case, of the students’ energetic responses to mathematical interaction. Central activities of ongoing analysis involve making local conjectures about students’ current ways of operating and designing new problems and problem sequences for the next teaching episode.

In conducting my ongoing analysis, I kept a research journal to record my impressions of each teaching episode, including my assessment of fluctuations in my own subjective vitality. The journal also provided a trace of decisions regarding changes in problems or approaches that I used with the students. In the weeks that I was not teaching, I reviewed the videorecordings and took additional notes on the preceding four to six episodes with each pair of students. Finally, and perhaps most important, I met regularly with the witness–researchers to gain their perspective on both the learning and caring aspects of the episodes. Discussions with them about the students’ current ways of operating, ways of operating that may have been within the students’ ZPCs, and the students’ energetic responses to our interactions, were highly valuable for me in preparing for subsequent teaching episodes, in formulating working models of students’ mathematics, and in achieving working interpretations of how I was (or was not) caring mathematically for students.

Retrospective

The main goal of retrospective analysis is to build analytic models of students’ mathematical thinking. These models are explanatory in the sense that they provide an account of the students’ ways and means of operating that, if followed, should produce the same states and ways of acting as observed in the students. In my use of the methodology, I also aimed to formulate an analysis of the energetic responses of the students and the teacher over the corpus of interactions, and to explicate themes in the establishment of MCRs with the students.

I began intensive retrospective analysis of the videorecordings by reviewing all videos chronologically with the purpose of making, testing, and revising conjectures (Cobb & Whitenack, 1996). My notes in this phase focused on these five categories: accounts of what students were able to do mathematically; changes in
how students operated; constraints that I experienced with the students; instances in which students indicated that they received my mathematical care; and instances in which students did not seem to receive mathematical care. My main criteria for the latter two categories were my assessments that students appeared to be experiencing (a) maintenance or enhancement of subjective vitality; (b) some depletion that was not severe or prolonged; or (c) significant depletion. I used all five categories to identify critical segments within teaching episodes on which I would focus deeper analysis.

Once I had identified critical segments, I analyzed the data with the goal of making analytic models of the students’ mental activity (the usual goal of the methodology). Then, to understand MCRs I did a second analysis from the point of view of the student–teacher pair. My goals were to uncover chains of student–teacher interactions that contributed to and detracted from the establishment of an MCR with each student. In this second analysis, I made passes through the data from two perspectives. First, I categorized chains of interaction according to when my actions as a teacher seemed to harmonize with a student’s mathematical thinking and energetic responses; when my actions appeared to trigger mathematical activity that seemed productive in some way for the student (e.g., that led to an accommodation); when I responded to a student in a way that did not seem productive for the student (cognitively and/or energetically); and when I recalibrated my response (intentionally or not) in an interaction with a student.

Second, for each chain of interaction I coded students’ and the teacher’s body language, facial expressions, verbal expressions, and verbal tone, to develop a more fine-grained picture of when students and the teacher seemed to experience enhancement or depletion of subjective vitality in our interactions. For example, when coding a particular chain of interaction, the body language code putting hand to head for a student was often accompanied by codes such as verbal expression of bafflement, teacher challenging students’ thinking and/or activity, and teacher searching for a task to pose. Any one of these four codes could indicate that the student was experiencing depletion during that segment of our interaction, and that the teacher may have been as well. All four of them together strengthened the conclusion about the student’s energy level in relation to my activity and energy level as a teacher. Based on these codes, I wrote analytic memos that would serve as the foundation for telling a story of MCRs from the data.

Finally, I also consulted my research journal for my assessment of fluctuations in my own subjective vitality, and my conjectures about students’ energetic responses, at the time of the experiment. For example, in the case above, my journal may have revealed that I felt particularly baffled and depleted by a student’s struggles but then felt buoyed by the end of the episode because the student seemed to be working more fluidly and her depletion seemed alleviated. The impressions I had at the time of the interaction were not always the conclusions I drew during retrospective analysis. For example, although I may have believed at the time of the interaction that a student’s depletion was alleviated relatively quickly, upon retrospective analysis I might conclude that it was sustained over a longer period. Both
my impressions at the time and my conclusions during retrospective analysis informed my construction of the cases.

ESTABLISHING MATHEMATICAL CARING RELATIONS:
TWO ILLUSTRATIVE EXAMPLES

In this section, I describe and analyze the establishment of MCRs with two students, Michael and Bridget, who were not paired. I focus on these two students because they highlight contrasting and similar characteristics of establishing MCRs. For example, throughout the year Michael and I established an MCR relatively steadily, whereas Bridget and I did not. However, in both situations I had to learn new mathematical ways of thinking to maintain or reestablish our MCR. The excerpts of data that I discuss—from one episode in February with Michael and four episodes in May with Bridget—are representative of the interactions with each student over the entire experiment. I have chosen them because they demonstrate important perturbations that the students—and I—experienced while working on RMR problems.

Michael: Harmonizing and Challenging in February

Making the New Collection of Candy Bars

On February 18, Michael’s partner, Carlos, was absent for the second time during the teaching experiment. So, Michael attended the session alone. Midway through the episode I asked Michael to make seven inch-long candy bars in the microworld, which he did (Figure 1). Then I posed the following RMR problem:

The Candy Bar Problem. That collection of 7 inch-long candy bars is 3/5 of another collection. Could you make the other collection of bars and find its total length?

The following data excerpt shows Michael’s initial response to this problem, as well as how he, I, and a witness–researcher interacted at the start of his work.

Excerpt 1: Michael’s perturbation in making the new collection of bars on February 18.

M: Okay, there are seven so . . . [He aligns the seven bars into one row.] I’m just doing this to try something out—

T: [Quickly] Yeah, that’s fine. This is a pretty hard problem, so we’re going to have to think about this a little bit.

M: [Completing the alignment of the bars, under his breath] Three-fifths. [Aloud] So something should be able to divide in three. [There is an 8-second pause while he

10 In the data excerpts, M stands for Michael, B for Bridget, T for the teacher–researcher (the author), and W for a witness–researcher. Comments enclosed in brackets describe students’ nonverbal action or interaction from the teacher–researcher’s perspective. Ellipses ( . . . ) indicate a sentence or idea that seems to trail off. Four periods ( . . . . ) denote omitted dialogue.
tracer over the 7 bars with the mouse. Two wouldn’t work, ’cause it’s odd [laughs a little]. Have to divide ’em into something.

T: Oh okay, all right.

M: There’s seven there, so. [He dials to 3 in the Parts menu and partitions each of the 7 inch-long bars into three equal parts, which I’ll call mini-parts.] Three-fifths . . . [He adjusts each partitioned bar slightly, realigning them with the mouse.]

T: Yeah, so that’s three fifths of another collection.

W: You want him to make the other collection?

T: Mm-hmm [yes].

M: There’s twenty-one [parts] there anyway, divided by three would be seven.

Excerpt 1 demonstrates that Michael had no immediate way of operating to solve the Candy Bar Problem. His comments about dividing “in three” combined with his activity of partitioning each of the seven bars into three equal parts provide evidence that he formed a goal to divide the 7 inches into three equal parts (corroborated by his later activity that I describe subsequently). However, he was not sure how to achieve that goal—and so I infer that he entered a state of perturbation. In particular, seven seemed to be a perturbing element for him.

Following this data excerpt, Michael made a 7-mini-part bar from the row of 21 parts, and he aligned it beneath the remaining mini-parts (Figure 2). Under his breath, he counted the number of mini-parts in the top row that were unmatched with the lower 7-mini-part bar (i.e., starting with the eighth mini-part in the top row). Upon determining that there were seven unmatched mini-parts, he announced excitedly “I did it!” Thus, he appeared to confirm that he had indeed transformed the original seven bars into three equal parts—a result he had likely expected and was learning to anticipate (Hackenberg, in press). Then, prior to carrying out further activity, he said, “And I know how many there are now.” He made the new collection of candy bars by repeating one of the 7-mini-part bars to make two more 7-mini-part bars, and he noted that there were “35 in all” (Figure 3).

Analysis: Constructing a New Scheme

Elsewhere I present a detailed analysis of Michael’s construction of a new scheme during this episode (Hackenberg, in press). Here I present an abridged version of
Michael’s initial goal to split a composite unit of seven into three equal parts indicates that he was using his reversible fraction scheme to solve the problem. That is, given a quantity that was three fifths of another quantity, Michael aimed to partition the given quantity into three equal parts, each of which would be one fifth of the other quantity and could be iterated five times to make the other quantity. However, he was not immediately certain of how to accomplish this goal. His insight was to insert units within each of the seven units to convert the composite unit of 7 inches into a composite unit consisting of a number of parts (21) that he could reorganize, in thought, as a unit of three units each containing seven units. In other words, he used the units-coordinating activity of his multiplying scheme (Steffe, 1992) to accomplish his initial goal.

I infer that he could operate in this way because he could view the 7 inches as a unit of seven units, into each of which he could insert more units (three units) to produce a number of units (21) that he could reorganize, in thought, as a unit of three units each containing seven units. Once he had made one fifth of the new collection, he could iterate the result (the 7-mini-part bar) to make the entire new collection. So, in solving the Candy Bar Problem, Michael inserted the coordination of units (the activity of his multiplying scheme) into the activity of his reversible fraction scheme. This way of operating was novel for Michael, and during the rest of the study he used this new scheme to solve other problems similar to the Candy Bar Problem (Hackenberg, 2005a, in press).
Determining the Length of the New Collection

The continuation of Excerpt 1 occurred directly following Michael’s comment about making “35 in all.”

Continuation of Excerpt 1: Michael determining the length on February 18.

T: Thirty-five—now thirty-five what—what are those little pieces?

M: Thirty-five . . . sevenths.

T: Thirty-five sevenths—is that what this little piece is of the unit bar? [The teacher points to one of the mini-parts, meaning to ask whether each mini-part is 1/7 of the unit bar.]

M: Wait! [Laughs a little.] That was twenty-one [referring to the original collection], so twenty-eight, and then thirty-five. But you said that [the original collection] would be three fifths so it [the new collection] would be five fifths.

T: Okay! So—

M: Thirty-five—wait—thirty-five—thirty fifths.

T: Thirty-five thirty fifths? Okay.

[A witness–researcher, who has come over to watch, shakes his head as if he can’t believe it, i.e., as a sign of being impressed by Michael, and turns away.]

T: [Acknowledging the witness–researcher’s impressed reaction, laughing in agreement.] You’re doing great Michael! But I wonder about, compared to the unit bar [pointing at the unit bar on the screen], what’s one of these little pieces [pointing at one of the mini-parts]? . . .

[After a 70-second discussion with the witness–researcher about how Michael made the new collection, the teacher and Michael return to the question of length.]

T: I wonder how long the collection is now?

M: It would be . . . three can’t go into thirty-five equally, so it would have to be a fraction, so . . . it would have to be eleven and two thirds.

T: How’d you get that so fast?

M: ’Cause three will go in there only eleven times because eleven times three equals thirty-three, and there’d be two left over, and since the unit bar would be divided by three, it would be two thirds.

T: Oh, my! Wow, Michael! [Both the teacher and the witness–researcher are impressed—they almost don’t know what to say.]

The continuation of Excerpt 1 demonstrates that Michael did not immediately know the length of the new collection he had made. Naming the new collection thirty-five sevenths may have come from Michael’s recent activity of joining mini-parts into bars that each consisted of seven mini-parts; creating 7-mini-part bars was an important part of solving the problem and could have led to his initial view...
of each mini-part as one seventh. His second name, thirty-five thirty-fifths, seemed to reflect having made five fifths of the new collection from the initial three fifths of it. That is, the new collection of bars was the whole for Michael when he thought about its length. However, after reviewing with me that the unit bar was 1 inch in length, he was reoriented to think in terms of inches, or at least in terms of unit bars. I infer that he knew he could not make a whole number of unit bars out of 35 mini-parts because he said, “Three can’t go into 35 equally.” Thus his response of 11 and 2/3 likely meant that the new collection was 11 unit bars and 2/3 of a unit bar.

Establishing an MCR With Michael

Michael’s solution of the Candy Bar Problem provides evidence of the evolving interactions between Michael and me that I characterize as mathematically caring. In fact, the February 18 episode was pivotal in establishing an MCR with Michael.\textsuperscript{11} Perhaps most prominent, his work on the Candy Bar Problem helped me understand the power of his ways of operating and the subjective vitality that he experienced during his mathematical activity.

The Candy Bar Problem as an act of mathematical care. Posing the Candy Bar Problem was not my intention for that episode because I was not sure that such a problem was within Carlos’s ZPC. So, Carlos’s absence from school that day, of which I was unaware until I went to escort the boys from their classrooms, meant I had an opportunity to adapt my plans in order to investigate my conjectures about Michael’s ways of operating. I had already observed that Michael could make various coordinations of units within units, demonstrating evidence of a reversible fraction scheme with composite units, an iterative fraction scheme, and the coordination of two fractions within the same bar (Hackenberg, 2007, in press). At the time I did not fully understand how closely related these ways of operating were, but that does not diminish the fact that my observations of them influenced my decision to pose the Candy Bar Problem to Michael when I had the opportunity. Making this adaptation in the moment involved both intuitive responsiveness to and analysis of Michael’s mathematical ways of operating, and it is an example of my own cognitive decentering.

Still, in the moment I was not sure how Michael would handle the Candy Bar Problem. Somewhat to my surprise, the problem provided a situation that seemed to be in harmony with his current schemes and operations. Michael used his reversible fraction scheme in assimilating the problem. But the problem also seemed to challenge him to modify his scheme. The challenge came from the opportunity to coordinate operations from his reversible fraction scheme and multiplying scheme in constructing a new scheme. So, the Candy Bar Problem was within Michael’s ZPC. Furthermore, because he greeted this challenge with interest and energy, as

\textsuperscript{11}It may seem contradictory to call the episode both representative and pivotal. It was representative of Michael’s overall engagement in teaching episodes throughout the experiment; it was pivotal for reasons I describe in this section.
I will now analyze, I can conclude that posing the problem was an act of mathematical care for him.

Receiving and giving mathematical care: Energetic responses to perturbations. In solving the Candy Bar Problem, I infer that Michael experienced at least two perturbations. The first was likely the most significant one, when I initially posed the problem to him and he aimed to divide the seven bars into three equal parts. I have already indicated how he eliminated this perturbation by coordinating the activity of his multiplying scheme with his reversible fraction scheme. Now I focus on his affective manner during this perturbation. His response, as shown in Excerpt 1, was exploratory, in that he began to move the bars in the microworld. He also seemed slightly nervous, in that he communicated that he was “just” trying something out, lest I think or expect that he knew exactly what to do. His small laugh upon rejecting the idea of “two” as a solution path corroborates a sense of nervousness and uncertainty. But above all, during the perturbation Michael appeared attentive and thoughtful, as if he was in a state of high energy and relatively low tension. In other words, he demonstrated calm energy (Thayer, 2001), one manifestation of the experience of subjective vitality (Ryan & Deci, 2008).

My response to Michael’s perturbation also indicates some evidence of subjective vitality. Since I was exploring to find out what Michael might do in the situation, I was watchful and curious, open to what he would do but also ready to abandon the problem if it did not seem to be fruitful for him. I also aimed to be reassuring in response to his apparent nervousness or uncertainty by telling him that the problem was challenging and by accepting his exploratory activity. So I was also in a state of calm energy, both observant and responsive. Being observant and responsive as a teacher in this kind of teaching experiment is not unusual (Cobb & Steffe, 1983; Steffe & Thompson, 2000b). But examining the energetic aspects of the teacher’s manner is.

In the second perturbation, in which Michael was faced with determining the length of the new collection, Michael’s and my energetic responses were slightly different. Michael’s manner was a bit quicker and less exploratory, perhaps largely due to the nature of the problem at this point (i.e., finding the length was more of a closed task than making the new collection had been). He was still thoughtful and attentive, but somewhat more eager than during the first perturbation. Another reason for this change could have stemmed from my responses and the responses of the witness–researcher during the continuation of Excerpt 1. From our verbal praise as well as our pleased and somewhat surprised attitudes, Michael seemed to be able to tell that what he was doing was impressive to us and that we held a high opinion of his work. The experience that his work was deemed impressive likely enhanced

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12 Given Michael’s subsequent activity, “Two wouldn’t work, ’cause it’s odd” may have meant partitioning each of the seven bars into 2 equal parts. That is, he could have anticipated producing 14 parts from doing so, and rejected the idea that 14 parts could be divided easily into 3 equal parts. However, it is not clear whether by “odd” he meant the initial seven bars or the endeavor to make them into 3 equal parts.
his subjective vitality, since he seemed pleased that he was doing well in our eyes. In short, he demonstrated a more obvious sense of enjoyment and perhaps a more aroused state than “calm” implies (Pekrun, 2006). His manner still supports the inference that he was experiencing subjective vitality in the interaction.

In turn, as shown in the continuation of Excerpt 1, I continued to attend closely to Michael’s activity. But in some ways I was less open—because I was more certain that challenging him to complete the entire problem was within his ZPC and seemed to be engaging to him. In other words, during the time in which he made the new collection of bars, I developed a stronger belief that he could determine its length. So I aimed to encourage his attainment of this goal. My energetic state was also somewhat more “excited,” which is clearly shown in my overt praise for Michael and my persistence in asking questions that oriented him but did not dominate his independent contributions to our interaction. In this interaction I was acting as a teacher–researcher normally would in this kind of teaching experiment: trying to test the limits of his reasoning in order to formulate an understanding of the boundaries of his ZPC, given that in making the new collection he had already operated in a way that had (pleasantly) surprised me. But equally important, from a perspective of a teacher–researcher working to establish an MCR, I wanted to assist Michael in sustaining or augmenting his experience of subjective vitality in this situation. Opening possibilities for him to use the full power of his reasoning to solve a problem he had not previously solved, thereby potentially expanding his mathematical world, was one way to do that.

The Candy Bar Problem episode as a perturbation for the teacher. Upon Michael’s solution of the Candy Bar Problem, I experienced pleasant surprise from the confirmation of my informed hunch that the problem would be within Michael’s ZPC. In a broader sense, our interaction during the episode provoked a perturbation for me in two ways. First, the episode pushed me to learn more about how to harmonize with Michael’s schemes and energetic responses and to envision new possibilities for challenging him to modify his schemes and build on his experience of subjective vitality. At the time of the experiment (in ongoing analysis), I made an assessment that solving RMR problems in which the initial quantity was a fraction was within Michael’s ZPC (e.g., a candy bar is 2/3 of a meter long, and that is 3/5 of the length of another candy bar). I also conjectured that he was ready to work toward more algebraic solutions of RMR problems. So the episode was pivotal in advancing my working model of Michael’s mathematical thinking. In addition, the episode helped me develop my model of Michael’s emotional and energetic responses, such as my deepening understanding of the satisfaction he derived from his own mathematical activity, even during a perturbation.

Second, the episode set off a perturbation regarding my construction of myself as someone who is aware of her ability to communicate about mathematics with others, one aspect of personal teaching efficacy (Bandura, 1977; Tshannen-Moran, Woolfolk Hoy, & Hoy, 1998). Over the course of the experiment, but marked notably by this episode and those following, I developed a feeling of being able to
communicate well mathematically with at least one student: Michael. Although I was an experienced middle school and high school mathematics teacher, I had left classroom mathematics teaching because I believed I did not know well enough what my students were learning mathematically, or what they should or could learn. Establishing an MCR with Michael allowed me to reclaim and deepen my construction of myself as someone who could communicate mathematically with students in productive ways.

The Candy Bar Problem episode as a perturbation for Michael. The February 18 episode also appeared to set off a broader perturbation for Michael. Michael tended to consider problems thoughtfully and to make observations about aspects of problems that interested him and that went beyond what I had specifically asked. At this point in the teaching experiment, he had already expressed enthusiasm for working in JavaBars. He seemed to derive pleasure from the “fitting together” both of bars in the microworld and of his ideas. So, up to this point in the teaching experiment, he seemed to like me and like mathematics well enough to participate in the project.

However, on his initial information sheet in November, in response to the prompt “One thing that interests or excites me about this project is . . . ,” he wrote, “having a partner.” He and Carlos got along well, and Michael tended to be a little shy or at least quiet with the research team. An example of Michael’s preference for coming to our sessions with Carlos occurred during the episode on February 2, the first time Carlos was absent. Michael seemed a bit reluctant to come that day and somewhat uncomfortable being the only child in the room. So prior to the episode on February 18, Michael provided indication that he participated largely because he liked being with Carlos and not necessarily because of particular confidence in himself as a doer of mathematics or as an able mathematical thinker.

After this point, a shift occurred. Michael seemed to become more aware that we sometimes had to slow down for Carlos and that in doing so, Michael did not always have an opportunity to operate as he had on February 18, in the ways that were so pleasing to him. In general Michael was highly patient with his friend and did not blurt out responses or indicate frustration; in fact, he often used the wait time to reconsider his own ideas and search for more efficient ways to approach problems, another byproduct of the growing power of his schemes and his personal tendencies. But Michael’s motivation to attend the sessions seemed to become rooted more deeply in his enjoyment of his mathematical activity and to depend less on the presence of Carlos. So, this episode seemed to mark a shift in Michael’s mathematical self-concept—that is, his beliefs about his abilities to do and learn mathematics (Craven, Marsh, & Burnett, 2003; Davis, 2001; Marsh & Shavelson, 1985).14

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13 Personal teaching efficacy is considered to be perceptions and beliefs about one’s ability to communicate with a wide range of students (Tshannen-Moran, Woolfolk Hoy, & Hoy, 1998).

14 A central finding of research on self-concepts is that self-concept is less useful as a general descriptor and instead has multiple facets that are linked to specific social or academic domains. For example, researchers have found reliable measures of students’ mathematical self-concepts that are distinct from students’ language self-concepts (Marsh & Shavelson, 1985).
In contrast to the relatively steady and progressive establishment of an MCR with Michael, establishing an MCR with Bridget was persistently difficult throughout the teaching experiment. She and I would establish an MCR and then it would weaken, sometimes severely, before I could determine a way to reestablish it. A good example of this phenomenon occurred in May, in the last four teaching episodes of the experiment. Because the presentation of data with Bridget spans four episodes, the structure of the presentation differs from the structure of the presentation with Michael. Notably I describe key events on May 3 and May 10 prior to analyzing the May 12 episode in detail.

A Nadir on May 3

Bridget and I reached a nadir on May 3, when Bridget was unable to operate in solving a variety of problems and seemed to shut down emotionally. That day I had initiated a “problem-solving workshop” because of my conjecture that Bridget and her partner Deborah were feeling constrained in some ways and needed to explore problems in whatever ways they chose (e.g., with or without the use of JavaBars, with or without drawings). May 3 was also the first time the girls worked on separate computers, which we continued throughout May.

The problems I posed at the start of the May 3 teaching episode involved improper fractions and the multiplicative coordination of two different quantities, such as: “A runner runs a mile in 6 minutes; find how long will it take him to run 13/8 of a mile.” I now infer that these problems were outside of Bridget’s ZPC. Although I understood very soon that Bridget was struggling, in the moment I did not know how to adapt the problems so that she could act more independently, as well as feel more autonomous and in control. I resorted to a good deal of coaching, which was laborious for both of us. Toward the end of the episode, I posed problems that Bridget could solve using her current schemes, and her mood seemed to lift. But afterward I felt worn out, both tense and tired.

Our interactions in this episode taught me that a weakened MCR with a student can put a heavy burden on the teacher as well as the student. From my perspective and that of the witness–researcher, during the episode with Bridget I was empathetic, patient, and persistent. I decentered to receive Bridget’s emotional experience of darkness and frustration, as well as her cognitive experience of not being able to operate. I also endeavored to pose questions about the problem situation that might make it possible for her to operate and feel less frustrated. But I did not receive fully enough her specifically mathematical, cognitive experience of being bothered by finding 1/8 of 6 minutes, let alone finding 13/8 of it. So, my suggestions were not very effective for her and indicated that I did not decenter enough cognitively. Thus our interactions did not help her to alleviate her depletion, and the longer she remained in a depleted state, the more depleted I felt!

My interaction with Bridget on May 3 set off a rather fervent search for better ways to communicate with her mathematically: I was in a consciously conflictive
state of perturbation. That the teaching experiment was nearly over only augmented my discomfort. I felt abashed that I was having such difficulty harmonizing with Bridget’s thinking and energetic responses at this late date. Out of that search came a gradual reestablishment of an MCR over the final three episodes of the teaching experiment. I began to take more seriously that certain numbers bothered Bridget—particularly improper fractions. And I began to realize that Bridget and her partner Deborah were operating quite differently when solving problems with reversible multiplicative reasoning.

*Special RMR Problems for Bridget on May 10: Known Quantities Are Unit Fractions*

Another goal for the girls during May was to work on RMR problems like the Candy Bar Problem,\(^{15}\) as well as RMR problems in which the known quantity was a fractional amount rather than a whole number. Near the end of the episode on May 5, Deborah solved a problem similar to the Candy Bar Problem, whereas Bridget did not. Bridget’s glum demeanor during her work on that problem, combined with her experience of sustained depletion on May 3, led me to differentiate instruction for the girls during the last two teaching episodes on May 10 and May 12.

In each of these two episodes I posed a series of RMR problems whose context was a homemade race-car contest between two sixth-grade science classes. On May 10, using JavaBars, Bridget solved a few problems similar to the following, in which the known quantity is a unit fraction:

The Race Car Problem 1. One half of a meter is 2/3 of the distance the other homemade race car traveled. Can you make how far the car traveled and tell how far it went?

I felt encouraged, because Bridget seemed to be operating independently and relatively fluidly on her own computer.

However, toward the end of the May 10 episode, she experienced difficulty solving the following “easier” RMR problem\(^{16}\) with the constraint that she not make it in JavaBars:

The Water Problem. A liter is 1000 grams of water (1000 cubic centimeters). That’s 2/3 of the water in another container. How much water is in the other container?

The Water Problem is potentially easier than Race Car Problem 1 because the known quantity is a whole number: Partitioning 1000 grams into two equal parts may be easier than partitioning 1/2 meter into two equal parts, in the sense that

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\(^{15}\)Recall the Candy Bar Problem: That collection of 7 inch-long candy bars is 3/5 of another collection. Could you make the other collection of bars and find its total length?

\(^{16}\)The word “easier” is in quotes to indicate it is from my perspective based on my analysis of the conceptual operations involved in the problem, not from Bridget’s perspective.
doing the latter requires determining the size of those parts in relation to the whole meter. That is, solving Race Car Problem 1 requires a student to (mentally or materially) partition the “other” half meter into two parts, determine that there are a total of four equal parts making up the meter, and therefore conclude that each part is 1/4 of a meter. Steffe (2003) calls the operation involved in this activity recursive partitioning.

Bridget had solved problems similar to the Water Problem, albeit with smaller known quantities (e.g., the Money Problem), in her selection interview in October; in the first episode of the experiment; and at other times during the study. That she could not solve the Water Problem on May 10 until she made a drawing in the microworld was revealing. I conjectured that her current scheme for solving these problems relied on figurative material for its implementation. That is, I conjectured that she had not yet interiorized\(^{17}\) the operations she was enacting on the drawings in JavaBars. So, for the next teaching episode on May 12, I set about designing a sequence of tasks that might engender this interiorization.

A New Sequence of Tasks on May 12

My plan was to pose several problems with unit fraction quantities and the same fraction relationship. I would encourage Bridget to solve the first few with JavaBars and then ask her to solve one without making bars. We went through three rounds of this sequence of problems, first with the relationship two fifths between known and unknown quantity, then with the relationship three fourths, and finally with four fifths. In round one, with the relationship two fifths, the following was the third problem that I posed to her:

The Race Car Problem 2. The Lizards’ car travels 1/7 of a meter. That’s 2/5 of the distance the Cobras’ car travelled. Can you make how far the Cobras’ car travelled and tell how far it went?

Bridget partitioned a copy of the unit meter into seven equal parts. Then she copied this 7/7-bar, partitioned each seventh into two parts and colored the first five parts (see Figure 4).

Excerpt 2: Bridget’s solutions of race car problems on May 12.

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\(^{17}\) If a scheme is interiorized, its results can be taken as given in further operating because the operations of the scheme do not have to actually be implemented.
T: Wow. So I wonder if you could do this one in your head. [Bridget frowns and raises her eyebrows, glancing uncertainly at the teacher.] So this time the Lizards went one eleventh of a meter.

B: [Looking worried.] Mm-hmm!?

T: And that’s two fifths of what the Cobras went.

B: [Continues to look worried.] Mm-hmm!? 

T: Can you tell how far the Cobras went?

B: [A little plaintively] No.

T: Oh I bet you can. See if you can imagine it.

B: [Rubs her eye.] So they went one eleventh?

T: Mm-hmm [yes].


T: [Whispers excitedly.] How’d you get that?

B: You can divide each eleventh into two, and you have twenty-two, and you got to have five instead of two.

T: Awesome, awesome Bridget!

**Analysis of Bridget’s Solution**

I infer that Bridget used her reversible fraction scheme to solve Race Car Problem 2 and its variation shown in the data excerpt. That is, she knew that if she had a quantity that was two fifths of another quantity, then dividing it into two equal parts would produce fifths of the other quantity, and three more of those fifths were needed to make five fifths, the other quantity. Furthermore, she used recursive partitioning to determine the size of the parts she had created in relation to the unit meter. Her solution of the variation on Race Car Problem 2, starting with 1/11 of a meter, demonstrates that she could perform her solutions in visualized imagination, and thus that she may have begun interiorizing her operations. In fact, I infer that she was abstracting a pattern from her operations to solve RMR problems in which the known quantity was a unit fraction. Throughout this episode, Bridget
operated similarly with the other rounds of problems (with the relationship three
fourths between known and unknown quantity, and then with the relationship four
fifths). Thus, the episode may have marked her initial construction of a scheme for
solving RMR problems in which the known quantity is a unit fraction.\(^\text{18}\)

In addition, Bridget was aware, to some extent, of the nature of the problems she
was solving and seemed to feel a degree of comfort and control from being able to
solve them. I make this claim because for the last problem of the episode, I
suggested that the Lizards’ car went three tenths of a meter. Bridget immediately
exclaimed, “Uh-oh, what?!” So she appeared to recognize that starting with three
tenths of a meter was quite a different problem. I stated that three tenths of a meter
was two fifths of the other distance. She began by making a 3/10-meter bar. But 90
seconds later she commented, “This is hard,” and after 60 more seconds she said,
“I don’t get it.” I suggested she think about what she had done when she had started
with, say, one third of a meter. She returned to that problem and told me that she
“took it [the 1/3-meter bar] out, divided it by two, and added three.”\(^\text{19}\) While I
interacted with Deborah, Bridget cleared the 3/10-meter bar, partitioned it into two
parts, pulled out one part, repeated it to make three parts, and joined the partitioned
3/10-bar and the 3-part bar (see Figure 5). When Bridget explained her work to me,
she said she knew that 4/5 of the Cobras’ distance was 6/10 of a meter (because 2/5
of the Cobras’ distance was 3/10 of a meter), but she did not know what five fifths
was. She seemed convinced that she had indeed made the Cobras’ distance, but she
did not know its measure in meters.

I praised Bridget’s work, but challenged her to figure out the length. By visually
inspecting the bars, she determined that she could divide each tenth of the 3/10-
meter bar into two equal parts to make six mini-parts, 15 of which made the Cobras’
distance. But she concluded that the Cobras’ distance was 15/19 of a meter because
she miscounted the number of mini-parts that would fit into the unit meter (rather
than relying on her recursive partitioning operation to determine the size of the
parts). So I infer that she was far from constructing a general scheme for solving
RMR problems with any fraction as the known quantity. A major constraint was
that she did not seem to have constructed a way to partition a 3-part bar into two
equal parts without relying on visual comparison and estimation. This way of
operating was exactly what Michael had successfully constructed in solving the
Candy Bar Problem.

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\(^\text{18}\) Because this episode was the last in the teaching experiment, I cannot conclude with any more
certainty that she had constructed such a scheme—I had no way of testing my conjecture in further
interaction with her.

\(^\text{19}\) Elsewhere (Hackenberg, 2007), I have suggested that this solution of adding on three parts, in
which each part was created by taking half of the 1/3-meter bar, is operationally quite different from a
solution in which a student iterates half of the 1/3-meter bar five times. The two solutions may indicate
differences in the students’ multiplicative concepts.
(Re-)Establishing an MCR With Bridget

The new sequence as an act of mathematical care. As I have described, the sequence of tasks I planned for Bridget on May 12 was a very deliberate attempt to harmonize with her schemes and operations and her energetic responses to our mathematical activity while challenging her to expand her ways of operating. In contrast with my interactions with Michael on February 18, on May 12 Bridget and I spent much of our time on problems that were well within her current ways of operating. Yet similar to my interactions with Michael, I continued (carefully) to challenge Bridget beyond what she seemed to imagine or think possible. In particular, when I asked her to solve a variation of Race Car Problem 2 without making the bars, I was trying to communicate to her that she could do something she thought she could not do. Of course, in the moment I did not myself know for sure that she could do it. I had conjectured that she could, but I also anticipated, based on her work on May 10, that asking her to do so could result in a significant perturbation for her. So I was testing in two ways. I wanted to see whether she could make the coordinations mentally, without activity in the microworld, in order to see whether she might interiorize her operations in these situations—a typical goal in this kind of teaching experiment. I was also testing whether she could weather the potential depletion from the perturbation (Tzur, 1995) and in the process maintain or even enhance her subjective vitality, a goal specific to my investigation of MCRs.

Bridget’s reception of mathematical care. As I have noted, my act of mathematical care would not indicate the reestablishment of an MCR with Bridget unless she experienced some alleviation of the depletion she had sustained over the three prior episodes. In fact, she responded quite positively to the sequence of tasks, both cognitively and affectively. I have already described the cognitive aspect: how I conjecture she began construction of a scheme for solving race car problems in

![Figure 5. The 3/10-meter bar (third bar from top, beneath the 10/10-meter bar), cleared of tenths marks and partitioned into two parts (fourth bar from top), joined with three more of those parts (bottom bar).](image-url)
which the known quantity was a unit fraction. This observation alone is a good sign that she could weather and eliminate perturbations she was experiencing. However, this observation alone does not provide the whole story, because her affective responses to my requests for her to solve race car problems in her imagination evolved dramatically over the episode. So, I posit that energetically she went through an interesting transformation.

At my first request that Bridget solve a problem without making the bars in the microworld, shown in Excerpt 2, Bridget looked worried: She frowned, raised her eyebrows, and spoke “no” plaintively. These behaviors all point to the experience of a perturbation—and accompanying tension over a concern that she might not be able to eliminate it. However, Bridget solved the problem in less than 15 seconds and justified her solution. My second request came about 4.5 minutes later, after she had solved, using JavaBars, two more such problems with the relationship three fourths between known and unknown quantity. I said, “I wonder if you could do this one by imagining it.” “Probably not,” she responded, which, at least on the surface, demonstrates doubt over meeting my request. Yet, she made no other protest and solved the problem, with justification, in approximately 15 seconds. Thus, this second request did not seem to engender the same perturbation for her, or the same level of tension, in the sense that she had some activity in which to engage, even though she did not express optimism about being able to solve the problem.

My third and final request came about 5 minutes later, after she had used JavaBars to solve two more problems with the relationship four fifths. “See if you can do this one in your head,” I said. Bridget said, “Okay.” Again she had a response in about 15 seconds but realized it was not quite correct and asked to do it on the computer. So, this third request appeared to provoke a perturbation for her—the problem was not transparent, even though she had a way of operating in which to engage. However, it did not provoke the same affective response—instead, it engendered activity in which she seemed to be interested. This change in her responses from unconfident and worried to agreeable and willing was a marked example of the alleviation of depletion, which I contend was occasioned by our interaction. That is, the formulation and sequencing of the problems in relation to my model of Bridget’s thinking allowed Bridget to engage in work that seemed sensible to her (an assessment I could determine only in retrospect, after posing the tasks in interaction with her). Working on tasks that seemed sensible to her—particularly after episodes in which the problems I posed to her were “weird” (her word)—appeared to be a relief, both cognitively and energetically.

An alternative interpretation is that experiencing mathematical success—that is, being able to solve the problems correctly—was the critical factor in the alleviation of her depletion. I find this second interpretation to be less complete when considering Bridget’s experience during the episode because she did not solve every problem “correctly” in visualized imagination and was not able to solve the last problem in which the given quantity was 3/10 of a meter. Yet she did successfully make sense of the problems that I was posing to her, even recognizing that the last one was somewhat different from the others. However, even if the experience of
mathematical success were the dominant factor, I maintain that this experience was open to her because our interaction allowed her to engage in activity that was sensible to her. Although perhaps Bridget did not reach a level of obvious enhancement of vitality similar to Michael, she was open and active and did not seem tired throughout the entire episode. In this way, Bridget received my mathematical care for her, and so cared back for me as a mathematics teacher.

The May episodes as a perturbation for the teacher. My response to Bridget’s initial perturbation on May 12 (as indicated in Excerpt 2) was similar to my response to Michael’s perturbations in two ways. First, I was in a heightened state of attentiveness to Bridget’s cognitive activity and energetic responses, especially because of the immediate history of our interactions. Second, as Michael had made the new collection and was determining its length, I felt some confidence that the new sequence of race car tasks was solvable by Bridget. With this confidence also came some tension, in that our MCR had eroded, and this was our last chance to interact in a more satisfying way during the experiment. So, Bridget’s evolving responses to the new sequence of problems, and the broader sense that we had reestablished an MCR, produced a significant boost in my own subjective vitality. I experienced relief—a definite decrease in tension. And I learned that when a student eliminates a perturbation, it can trigger the elimination of a perturbation for the teacher—at least locally.

In a broader sense, the May episodes triggered at least three significant perturbations for me. First, our interactions during May provoked me to see Bridget’s personal tendencies in a new light. Throughout the teaching experiment, I persisted in the view that Bridget had schemes nearly as powerful as those of her partner, Deborah, only Bridget seemed to lack confidence. Deborah tended to convey certainty about her own mathematical reasoning. In contrast, Bridget did not expect to understand everything nearly instantly: Her mathematical self-concept seemed to include the belief that she would not understand. I believed that her apparent lack of confidence meant that Bridget was not getting enough opportunities to express her ideas. This view was not completely unwarranted; Deborah’s swift and strong ways of operating could curtail Bridget’s mathematical activity and sometimes dampen her spirit (Hackenberg, 2005a). But my view of Bridget also blocked me from understanding that her seeming lack of confidence and her reticence were entwined with less powerful ways of operating with RMR problems. After the sustained depletion that Bridget experienced in early May, I felt driven to learn how to interact with her in a way that would open possibilities for her to be more operative, which we achieved on May 12.

However, that learning alone did not eliminate a second perturbation for me, because I aimed to understand how to account for Bridget’s difficulties and to imagine mathematical problems that would have been more appropriate for her during times in the experiment when she experienced depletion. Having arrived at a new insight in my working model of Bridget on the last day of the experiment, I was now challenged to develop an analytic model in retrospective analysis
(e.g., Hackenberg, 2007, in press). So, as with Michael, the May episodes led to changes in my model of Bridget’s mathematical thinking.

In addition, these four episodes with Bridget represented a third perturbation for me involving my personal teaching efficacy, perhaps more profoundly than my interactions with Michael. It was consistently gratifying to be able to trust that Michael and I could communicate mathematically. However, with Michael I was often not put to the test in the same way that I was with Bridget. Michael and I certainly had plenty of moments when we did not necessarily understand what the other person was saying or doing. But I had a feeling that I would be able to understand Michael’s mathematical activity eventually, and I had a sense that he trusted that the problems I posed would be interesting to him. He seemed to believe that I had his best interests in mind, even if he might not see that immediately. Although Bridget seemed to like me well enough, she did not seem to trust that the problems I posed to her would be interesting or sensible. In fact, throughout the experiment she accused me of posing “weird” problems to her. So it was very satisfying to find, on May 12, that I was learning to communicate better with a student whose ways of operating I did not always understand and who, for good reason, was suspicious of many of the problems that I posed to her. The ways in which our interactions in May were a larger perturbation for Bridget is not known, because the teaching experiment ended on May 12.

DISCUSSION

At the beginning of this article, I identified two questions as the focal points for this study: What is the nature of establishing MCRs in extensive, small-scale interaction between students and their teacher? and What is mathematical about the caring relations? In this section I respond to these questions. Then I consider three other questions raised by the study that could define directions for future research in this area. Finally, I suggest contributions that investigating MCRs makes to research on mathematical learning.

**Question 1: What Is the Nature of Establishing MCRs?**

Although establishing MCRs with Michael and with Bridget had some quite different characteristics, together the two cases illustrate three themes that were pervasive in the establishment of MCRs with all four students during the teaching experiment.

**Harmonizing and Challenging Are Linked**

As exemplified by the interactions between Bridget and me, difficulties in harmonizing with a student’s schemes and energetic responses likely preface difficulties in posing appropriate challenges. Furthermore, as exemplified by the interactions between Michael and me, posing challenges well (i.e., just on the edge of a student’s ZPC and without a significant or sustained drop in subjective vitality)
often signals harmonizing well with the student’s schemes and energetic responses. These linkages are entirely reasonable because the teacher’s model of a student’s schemes, operations, and energetic responses is the basis for formulating challenges for that student. So, if a teacher’s current model of a student includes some unwarranted assumptions (which are likely outside of the teacher’s current awareness), and the teacher formulates a challenge based on those assumptions, it is quite likely that the challenge will be inappropriate. The student may meet the challenge easily because she has well-developed operations or schemes that were not included in the teacher’s model, rendering the challenge ineffective in terms of expanding the student’s mathematical world. Alternatively, the student may not have constructed operations or schemes needed to meet the challenge, which may render the challenge overwhelming (at least cognitively, and perhaps energetically as well). Furthermore, if a student receives a challenge in a way that provokes an enhancement of subjective vitality, then the challenge likely is sensible to him or her—that is, the student can assimilate the challenge using current schemes and operations, even if the student does not immediately know how to solve a particular problem.

From a teacher’s perspective, the view that harmonizing and challenging are linked suggests an integrated perspective for assessing whether MCRs are being established, based on examining student responses to challenges. For example, if a teacher finds that students appear quite depleted upon participating in a particular mathematical activity, this can be a signal for the teacher to reassess both how to harmonize better with students’ schemes and energetic responses and then how to pose appropriate challenges. Similarly, if a teacher finds that students are appropriately challenged by a particular task, that can inform how he or she understands and harmonizes with students’ schemes and energetic responses in further mathematical activity. This study suggests that teachers who can link harmonizing and challenging in this way can build trust with students. That is, if students perceive the teacher as someone who can understand them and pose challenges that are “workable” and amenable, then they are likely to trust the teacher to help them learn—a fundamental expectation of both students and teachers.

The Linkage Between Student and Teacher Perturbations Is What the Teacher Aims to Influence

When students experience a perturbation, their cognitive and energetic responses to it can provoke a perturbation for the teacher. The teacher’s perturbation can, in turn, prompt action on the part of the teacher, which can influence the ways in which the students experience their perturbations. In particular, student–teacher interaction can be viewed as a linked “chain” of perturbations (Steffe, 1996). In student–teacher interaction aimed toward the establishment of MCRs, the linked chain tends toward perturbations that are bearable (Tzur, 1995) for both students and teachers. Because the teacher is the one monitoring this chain and aiming to influence it, the question becomes what tools the teacher has for doing so. One of the main tools of the teacher is taking both cognitive constraints and drops in subjective vitality seriously.
Cognitive constraints experienced with students, when not taken seriously, may be accompanied by energetic changes that can block opportunities for further growth. Similarly, drops in energy, when not taken seriously, may foreclose meeting a cognitive constraint with some forbearance. Taking cognitive constraints seriously requires cognitive decentering, as well as questioning one’s assumptions about students’ mathematical ways of operating. Taking drops in energy seriously means learning to assess what they likely indicate for a student, keeping open the possibility that they may indicate a cognitive constraint. My difficulty with harmonizing with Bridget’s schemes and energetic responses stemmed from assuming more than was warranted about her schemes, even though I persistently saw that her ways of operating were different from, and often seemed less powerful than, those of her partner, Deborah. This difficulty contributed to the chain of perturbations in early May that left us both depleted. Because Deborah could be quite dominating, and because Bridget sometimes seemed reticent, I tried to account for the differences between them by positing that Bridget did not have enough time to think or sufficient confidence in her own mathematical ideas. By making such explanations, I did not take seriously enough the cognitive constraints that both Bridget and I experienced in her ways of operating. This dynamic led to repeated weakening of our MCR over the course of the experiment and may have impeded progress she could have made in expanding her mathematical ways of operating.

**Acts of Learning May Be a Confirming Sign**

In the snapshot of the MCR with Michael on February 18, an act of learning occurred. Moreover, Michael’s learning was novel in that he had not previously encountered a problem like the Candy Bar Problem. Although prior tasks were not unrelated (Hackenberg, 2005a), I did not set up a careful sequence of tasks for him in posing the Candy Bar Problem. Thus, his construction of a new scheme was particularly striking. In the snapshot of the MCR with Bridget on May 12, I had carefully tailored and sequenced tasks to engender an act of learning, and it is possible that she was constructing a new scheme. However, this (potential) scheme solved a more limited set of problems than what I had originally intended or hoped for her. So, at least this much can be said: An MCR with a student may be established without the occurrence of any particular act of mathematical learning (Hackenberg, 2005b), although the construction of a new scheme may be a confirming sign of establishing an MCR.

**Question 2: What Is Mathematical About the Caring Relations?**

From the teacher’s point of view, establishing MCRs requires, above all, decentering from one’s own mathematical thinking and the construction of new mathematical ways of operating that fit with the teacher’s experience of the students. I have called this kind of decentering cognitive, in the sense that it requires a considerable amount of thinking. However, both analysis and intuitive responsiveness are part of cognitive decentering. The intuitive responsiveness “feels” much like
gaining insight when working on a difficult mathematical problem (Davis & Hersh, 1981): The exact pathways to such insight are not often clear at the time and not always traceable. Yet they usually involve cognitive activity in one of its most creative manifestations. The analytical aspect of cognitive decentering involves deliberate conjecturing about mathematical ways of operating that model the mathematical activity of the student. The new mathematical ways of operating, which the teacher uses to interact with the students, are pivotal for the teacher in establishing MCRs.

From the student’s point of view, these caring relations are mathematical because they foster responsiveness to the teacher through engagement in mathematical activity. That is, the student’s response to the teacher’s mathematical care is, in part, the student’s mathematical thinking and activity—evidence of this thinking and activity is part of what the teacher receives. The student’s mathematical thinking and activity are necessarily entwined with another part of what the teacher receives, the student’s subjective vitality.

Questions for Future Research

Although the data from this study allow me to respond to the study’s two focal questions, the study raises at least three additional questions that could define directions for future research in this area. First, how are MCRs a description of an interaction as opposed to a set of teaching behaviors?\(^{20}\) I have aimed for the former, but my account of the establishment of MCRs with two students is necessarily from my perspective as a teacher. Like many researchers, I have used my own teaching as a study site (e.g., Ball & Wilson, 1996; Chazan, 2000; Tzur, 2004). Although studying one’s own teaching is advantageous when developing a new construct, one disadvantage is the difficulty of sufficiently eliciting and portraying students’ perspectives. Future research could include teaching experiments that use written indicators for assessing mathematical care, such as measures of subjective vitality (Ryan & Frederick, 1997) or of the strength of student–teacher relationships (O’Conner & McCartney, 2007). For example, asking students to complete Ryan and Frederick’s 7-item subjective vitality measure periodically before or after teaching episodes could provide an additional index in assessing students’ energetic states. Another possibility is to invite retrospective reflection from students on some teaching episodes to explicitly elicit their perspectives on their cognitive and energetic responses to the mathematical interactions.

A second question is: How are the constructs developed in this study useful in establishing MCRs with a larger group of students? In other words, in more classroom-like settings, how does a teacher work to influence students’ subjective vitality while aiming for mathematical learning? It may be important for students to take on some of the mathematical caring responsibilities for fellow students. Although other students are not in a position to monitor or to seek to influence the chain of perturbations between teacher and student, they may be in a position to

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\(^{20}\) I thank an anonymous reviewer for suggesting this question.
support classmates in their inquiry process. For example, in this study, over time Michael demonstrated some mathematical care toward his partner Carlos in the following sense: When he could see that Carlos was uncertain or unable to proceed, Michael adopted my habit of asking “I wonder” questions. So, rather than just telling or showing Carlos what to do, Michael (often with a glance at me) would try to pose a question to Carlos that would help him continue to work on the situation. Michael and I never explicitly discussed this “strategy.” However, he seemed to understand that a main goal of our work together was not just completing problems, but building ideas about how to complete them, and he endeavored to help his friend toward that end. Future research on MCRs with groups of students could include studying the establishment (and scope) of norms with students that contribute to (or detract from) the establishment of mathematical care in a classroom.

In continuing to broaden the scope of this study, a third question emerges: How do MCRs fit into the larger landscape of caring for students in schools? In other words, how can the study of mathematical care and general care for students mutually inform each other? To respond to this question requires studying how MCRs develop in relation to other aspects of caring relationships in schools. However, it also requires addressing the critical idea that caring for students by teachers is not monolithic, but is influenced by students’ backgrounds, including their culture, race, gender, and socioeconomic status (Ding & Hall, 2007; Hayes, Ryan, & Zseller, 1994; Ladson-Billings, 1994; Thompson, 1998; Valenzuela, 1999). In short, MCRs need to be articulated in relation to the broader canvas of the development of care theory, which includes theorizing about care in relation to assumptions founded on particular racial and cultural points of view, as Thompson points out. So, future research on MCRs might explore their establishment in settings with students from diverse backgrounds, as well as settings in which students are from similar backgrounds, while continuing to question assumptions about theoretical formulations of care and caring in schools.

**Contributions of MCRs**

Acknowledging the need for further development of MCRs as a research area underscores Noddings’s (2001) point of view that insisting on a contribution from students to “complete” this particular form of student–teacher relationship, a caring relation, makes doing research on caring relations quite difficult. In other words, the study of dynamic and evolving relations, let alone the energetic aspects of those relations, is a challenge. But that does not mean that the research area should not be addressed (Williams, 2008).

In fact, this study indicates that the formulation and investigation of MCRs can make several contributions to research on mathematical learning. First, attending to subjective vitality in the process of aiming for mathematical learning opens possibilities for making and refining interventions so that both teachers and students may learn more. That is, monitoring energetic responses of students can
provide an impetus for teachers to reformulate models of students’ thinking and to recalibrate goals, problems, and activities that may be within (or just at the edge of) students’ ZPCs. Second, and reciprocally, learning better how to harmonize with students’ schemes and energetic responses, and how to challenge students to modify their schemes, can open possibilities for enhancing students’ subjective vitality—surely a worthy goal in and of itself.

Third, establishing MCRs validates the difficulties that both teachers and students may experience in eliminating significant perturbations. This contribution can be an important factor in developing appreciation for the often-protracted process of learning, and for learners (teachers as well as students) who engage in creating new mathematical ways of thinking. In addition, attending to both cognitive challenges and energetic fluctuations during the experience of perturbations can provide a way to better understand how student and teacher actions are linked.

Fourth, establishing MCRs appears to influence teachers’ personal teaching efficacy (Teven, 2007) and students’ construction of mathematical self-concepts (cf. Harter, 1996). Linking what one knows to one’s construction of oneself is not uncommon (e.g., Belenky, Clinchy, Goldberger, & Tarule, 1997; Boaler, 2000; Cobb, Gresalfi, & Hodge, 2009; Harding, 1991; Larochelle, 2000; Wenger, 1998; Wortham, 2004). This link is of considerable importance in mathematics education, because persistent feelings of inferiority in mathematical activity can lead to judgments about one’s ability to do mathematics, which may in turn engender broader, and often relatively permanent, judgments about whether one is, or is not, mathematical (Steffe, 2000). Similarly, persistent feelings of ineffectiveness in orchestrating mathematical activity for students can lead to judgments about one’s ability to communicate mathematically with them, which may in turn engender permanent judgments about oneself as a mathematics teacher.

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